Q9 Evaluate the Convolution of the two sequences

$$h(n) = (0.5)^n u(n) \quad \text{and} \quad x(n) = 3^n u(-n)$$

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \qquad |z| > \frac{1}{2}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{0} 3^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n z^n = \sum_{n=0}^{\infty} \left(\frac{1}{3}z\right)^n$$

$$= \frac{1}{1-\frac{1}{3}z} = \frac{1}{\frac{1}{3}z(3z^{-1}-1)} = \frac{3z^{-1}}{3z^{-1}-1}$$

$$= -\frac{3z^{-1}}{1-3z^{-1}} \qquad |z| < 3$$

$$y(n) = x(n) * h(n)$$

$$\therefore Y(z) = -\frac{3z^{-1}}{1-3z^{-1}} * \frac{1}{1-\frac{1}{2}z^{-1}} \qquad \frac{1}{2} < |z| < 3$$

$$Y(z) = \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1-3z^{-1}}$$

$$A = \left[\left(1-\frac{1}{2}z^{-1}\right) Y(z)\right]_{z=\frac{1}{2}} = \left.\cancel{\left(1-\frac{1}{2}z^{-1}\right)} * \frac{-3z^{-1}}{1-3z^{-1}} * \frac{1}{\cancel{1-\frac{1}{2}z^{-1}}}\right|_{z=\frac{1}{2}}$$

$$= \frac{-3\cancel{z^{-1}}}{\cancel{z^{-1}}(z-3)} = \frac{-3}{z-3} = \frac{-3}{\frac{1}{2}-3} = \frac{+3}{+\frac{5}{2}} = \frac{6}{5}$$

$$B = \left[(1-3z^{-1})\,Y(z)\right]_{z=3} = -\frac{6}{5}$$

$$y(n) = \left(\frac{6}{5}\right)\left(\frac{1}{2}\right)^n u(n) + \left(\frac{6}{5}\right) 3^n u(-n-1)$$

Q10 *Derivative property*

Find the Z-transform of $x(n) = |n|\left(\frac{1}{2}\right)^{|n|}$.

$$\left(\frac{1}{2}\right)^n u(n) \stackrel{Z}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}}$$

$$n\left(\frac{1}{2}\right)^n u(n) = -z\frac{d}{dz}\,\frac{1}{1-\frac{1}{2}z^{-1}} = -z * \frac{-1 * \frac{1}{2}z^{-2}}{(1-\frac{1}{2}z^{-1})^2}$$

$$= \frac{\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})^2}$$

$$x(n) = |n|\left(\frac{1}{2}\right)^{|n|} = n\left(\frac{1}{2}\right)^n u(n) - n\left(\frac{1}{2}\right)^{-n} u(-n)$$

Using linearity and the time-reversal property, we have

$$X(z) = \frac{\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})^2} + \frac{\frac{1}{2}z}{(1-\frac{1}{2}z)^2}$$

$$= \frac{\frac{5}{8}z + \frac{5}{8}z^{-1} - 1}{(1-\frac{1}{2}z^{-1})^2(1-\frac{1}{2}z)^2}$$

$$\frac{1}{2} < |z| < 2.$$

Q12 let $y(n)$ be a sequence that is generated from a sequence $x(n)$ as follows:

$$y(n) = \sum_{K=-\infty}^{n} K\,x(K)$$

a) show that $y(n)$ satisfies the time-varying difference equation

$$y(n) - y(n-1) = n\,x(n)$$

and show that

$$Y(z) = \frac{-z^2}{z-1}\,\frac{dX(z)}{dz}$$

where $X(z)$ and $Y(z)$ are the Z-transforms of $x(n)$ and $y(n)$, respectively.

b) Use this property to find the Z-transform of

$$y(n) = \sum_{K=0}^{n} K\left(\tfrac{1}{3}\right)^K \quad n \geq 0$$

a) From the definition of $y(n)$

$$y(n-1) = \sum_{K=-\infty}^{n-1} K\,x(K)$$

$$y(n) - y(n-1) = n\,x(n)$$

from the difference equation,

$$n\,x(n) \xleftrightarrow{z} -z\,\frac{dX(z)}{dz}$$

$$Y(z) - z^{-1}Y(z) = -z\,\frac{dX(z)}{dz}$$

or

$$Y(z) = \frac{-z}{1-z^{-1}} \frac{dX(z)}{dz} = \frac{-z^2}{z-1} \frac{dX(z)}{dz}$$

(b)

$$y(n) = \sum_{K=-\infty}^{n} K x(K)$$

$$x(n) = \left(\frac{1}{3}\right)^n u(n)$$

$$X(z) = \frac{1}{1-\frac{1}{3}z^{-1}} \qquad |z| > \frac{1}{3}$$

then $Y(z) = \dfrac{-z^2}{z-1} \dfrac{dX(z)}{dz} = \dfrac{-z^2}{z-1} \dfrac{-\frac{1}{3}z^{-2}}{(1-\frac{1}{3}z^{-1})^2}$

$$= \frac{\frac{1}{3}z^{-1}}{(1-\frac{1}{3}z^{-1})^2 (1-z^{-1})}$$

Because $x(n)$ is right-sided, then the region of Convergence is the exterior of a circle. Having poles at $z=1$ and $z=\frac{1}{3}$, it follows that the ROC $|z|>1$.

Q13 Find $x(0)$ for the sequence that has a Z-transform

$$X(z) = \frac{1}{1-az^{-1}} \qquad |z| > a$$

$$x(0) = \lim_{z\to\infty} \frac{1}{1-\frac{a}{z}} \Rightarrow x(0) = \lim \frac{1}{1-0} = 1$$

Q14 Find the value of $x(0)$ for the sequence that has a Z-transform

$$X(z) = \frac{z}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-2})} \qquad |z| > \frac{1}{2}$$

$$X(z) = \frac{z}{(1-\frac{1}{2}\frac{1}{z})(1-\frac{1}{3}\frac{1}{z^2})}$$

$$= \frac{z}{\left(\frac{2z-1}{2z}\right)\left(\frac{3z^2-1}{3z^2}\right)}$$

$$= \frac{z^4}{\left(\frac{2z-1}{2}\right)\left(\frac{3z^2-1}{3}\right)}$$

$$= \frac{z^4}{(z-\frac{1}{2})(z^2-\frac{1}{3})}$$

at $X(z) \to \infty \Rightarrow |z| \to \infty$

if we delay $x(n)$ by 1 to form the sequence

$y(n) = x(n-1)$

$$Y(z) = \frac{z^3}{(z-\frac{1}{2})(z^2-\frac{1}{3})}$$

which approaches 1 as $|z| \to \infty$, and that means

$y(0) = x(-1) = 1$. Because

$$X(z) = x(-1)z + \sum_{n=0}^{\infty} x(n) z^{-n}$$

$X(z) - x(-1)z$ is the Z-transform of a causal sequence, and it follows from the initial value theorem that

$$x(0) = \lim_{|z|\to\infty} \{X(z) - x(-1)z\}$$

$$X(z) - x(-1)z = X(z) - z = \frac{z^4}{(z-\frac{1}{2})(z^2-\frac{1}{3})} - z$$

$$= \frac{z^4 - z(z^3 - \frac{1}{2}z^2 - \frac{1}{3}z + \frac{1}{6})}{(z-\frac{1}{2})(z^2-\frac{1}{3})}$$

$$x(0) = \lim_{z\to\infty} \{X(z) - x(-1)z\}$$

$$= \lim_{z\to\infty} \frac{\cancel{z^4} - \cancel{z^4} + \frac{1}{2}z^3 + \frac{1}{3}z^2 - \frac{1}{6}z}{z^3 - \frac{1}{3}z - \frac{1}{2}z^2 + \frac{1}{6}}$$

$$= \lim_{z\to\infty} \frac{\cancel{z^3}\left(\frac{1}{2} + \frac{1}{3z} - \frac{1}{6z^2}\right)}{\cancel{z^3}\left(1 - \frac{1}{3z^2} - \frac{1}{2z} + \frac{1}{6z^3}\right)}$$

$$= \lim_{z\to\infty} \frac{\frac{1}{2} + 0 + 0}{1} = \frac{1}{2}$$

Q15 Generalize the initial value theorem to find the value of a causal sequence $x(n)$ at $n=1$, and find $x(1)$ when $X(z) = \dfrac{2+6z^{-1}}{4-2z^{-2}+13z^{-3}}$

If $x(n)$ is Causal

$$X(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots$$

if we subtract $x(0)$ from $X(z)$

$$X(z) - x(0) = x(1)z^{-1} + x(2)z^{-2} + \cdots$$

multiply both sides by z to get

$$z[X(z) - x(0)] = x(1) + x(2)z^{-1} + \cdots$$

$z \to \infty$, we obtain the value for $x(1)$,

$$x(1) = \lim_{|z|\to\infty} \{z[X(z) - x(0)]\}$$

we have

$$x(0) = \lim_{|z|\to\infty} X(z) = \frac{1}{2}$$

therefore, $X(z) - \frac{1}{2} = \dfrac{2+6z^{-1}}{4-2z^{-2}+13z^{-3}} - \dfrac{1}{2}$

$$= \frac{6z^{-1} + z^{-2} - \frac{13}{2}z^{-3}}{4-2z^{-2}+13z^{-3}} \quad \therefore$$

$$\therefore\ x(1) \simeq \lim_{|z|\to\infty} \{ z[X(z) - x(0)]\} = \frac{3}{2}$$

Q16 find $x(0)$ for $X(z) = \dfrac{3z^{-1} + 2z^{-2}}{3 - z^{-1} + z^{-2}}$

$$x(0) = \lim_{z\to 0} X(z) = \lim_{z\to 0} \frac{\cancel{z^{-2}}\{3z+2\}}{\cancel{z^{-2}}\{3z^{2} - z + 1\}} = \frac{2}{1} = 2$$

Q17 Use the derivative property to find the z-transform of the following sequences:

a) $x(n) = n\left(\frac{1}{2}\right)^{n} u(n-2)$

b) $x(n) = \frac{1}{2}(-2)^{-n} u(~~-~~n-1)$

a)

$$n\,x(n) \overset{z}{\longleftrightarrow} -z\frac{d}{dz}X(z)$$

if $x(n) = n\,w(n)$, where

$$w(n) = \left(\frac{1}{2}\right)^{n} u(n-2) = \frac{\left(\frac{1}{2}\right)^{n}}{4 \times \frac{1}{4}} u(n-2)$$

$$= \frac{\left(\frac{1}{2}\right)^{n}}{4 * \left(\frac{1}{2}\right)^{2}} u(n-2) = \frac{1}{4}\left(\frac{1}{2}\right)^{n-2} u(n-2)$$

from the delay property and the Z-transform pair

$$\alpha^{n} u(n) \overset{z}{\longleftrightarrow} \frac{1}{1-\alpha z^{-1}} \qquad |z| > |\alpha|$$

$$w(z) = \frac{\frac{1}{4} z^{-2}}{1-\frac{1}{2} z^{-1}} \qquad |z| > \frac{1}{2}$$

by using the derivative property,

$$x(z) = -z \frac{d}{dz} w(z) = -z \frac{(1-\frac{1}{2} z^{-1}) * \frac{1}{4} * -2 z^{-3} - \frac{1}{4} z^{-2} * \frac{1}{2} z^{-2}}{(1-2z^{-1})^2}$$

$$x(z) = -z \frac{[-\frac{1}{2} z^{-3} + \frac{1}{4} z^{-4} - \frac{1}{8} z^{-4}]}{(1-2z^{-1})^2}$$

$$x(z) = -z \frac{[-\frac{1}{2} z^{-3} + \frac{1}{8} z^{-4}]}{(1-2z^{-1})^2}$$

$$x(z) = \frac{1}{2} z^{-2} \frac{(1-\frac{1}{4} z^{-1})}{(1-2z^{-1})^2}$$

b) in this case, we have n^{-1}, we will define a new sequence, $y(n)$

$$\left(\frac{-1}{2}\right)^n \qquad \frac{1}{1-(-\frac{1}{2})z^{-1}}$$

$$y(n) = n x(n) = (-2)^{-n} u(-n-1)$$

$$Y(z) = \frac{-1}{1+\frac{1}{2} z^{-1}} \qquad |z| < \frac{1}{2}$$

$$+z \frac{d}{dz} X(z) = \frac{+1}{1+\frac{1}{2} z^{-1}}$$

$$\frac{d}{dz} X(z) = \frac{1}{z(1+\frac{1}{2} z^{-1})} = \frac{1}{z+\frac{1}{2}}$$

$$\therefore \frac{d}{dz} X(z) = \frac{1}{z+\frac{1}{2}} \Rightarrow dX(z) = \frac{1}{z+\frac{1}{2}}\, dz$$

$$\int dX(z) = \int \frac{1}{z+\frac{1}{2}}\, dz$$

$$X(z) = \ln\left(z+\frac{1}{2}\right) \qquad \text{ROC } |z| < \frac{1}{2}$$

Q18 Find the Z-transform of the sequence

$$x(n) = \begin{cases} \alpha^{n/10} & n = 0, 10, 20, \ldots \\ 0 & \text{else} \end{cases}$$

where $|\alpha| < 1$.

$$\alpha^n u(n) \xleftrightarrow{Z} \frac{1}{1-\alpha z^{-1}} \qquad |z| > \alpha$$

the Z-transform of $x(n)$ is

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n\times 10}$$

$$= \sum_{n=-\infty}^{\infty} x(n) (z^{10})^{-n} = X(z^{10})$$

$$\therefore X(z) = \frac{1}{1-\alpha z^{-10}} \qquad |z| > \alpha^{1/10}$$

Q19 Find the inverse of each of the following Z-transform

a) $X(z) = 4 + 3(z^{2} + z^{-2}) \qquad 0 < |z| < \infty$

b) $X(z) = \dfrac{1}{1-\frac{1}{2}z^{-1}} + \dfrac{3}{1-\frac{1}{3}z^{-1}} \qquad |z| > \frac{1}{2}$

c) $X(z) = \dfrac{1}{1+3z^{-1}+2z^{-2}} \qquad |z| > 2$

d) $X(z) = \dfrac{1}{(1-z^{-1})(1-z^{-2})} \qquad |z| > 1$

b) $x(n) = \left(\frac{1}{2}\right)^{n} u(n) + 3\left(\frac{1}{3}\right)^{n} u(n)$

c) $X(z) = \dfrac{1}{(1+2z^{-1})(1+z^{-1})}$

$= \dfrac{A}{1+2z^{-1}} + \dfrac{B}{1+z^{-1}}$

$A = \cancel{(1+2z^{-1})} * \dfrac{1}{\cancel{(1+2z^{-1})}(1+z^{-1})}\Bigg|_{z=-2}$

$A = \dfrac{z}{z+1}\Bigg|_{z=-2} \Rightarrow A = \dfrac{-2}{-2+1} = \dfrac{-2}{-1} = 2$

$$B = \cancel{(1+z^{-1})} * \frac{1}{(1+2z^{-1})\cancel{(1+z^{-1})}}\Bigg|_{z=-1}$$

$$B = \frac{z}{z+2}\Bigg|_{z=-1} \Rightarrow B = \frac{-1}{-1+2} = \frac{-1}{1} = -1$$

$$\therefore X(z) = \frac{2}{1+2z^{-1}} - \frac{1}{1+z^{-1}}$$

Because $x(n)$ is right-sided, the inverse Z-transform is

$$x(n) = 2(-2)^n u(n) - (-1)^n u(n)$$

d) $$X(z) = \frac{1}{(1-z^{-1})^2(1+z^{-1})} = \frac{A}{1+z^{-1}} + \frac{B}{1-z^{-1}} + \frac{C}{(1-z^{-1})^2}$$

$$\frac{1}{(1-z^{-1})(1-z^{-1})(1+z^{-1})}$$

$$A = \cancel{(1+z^{-1})} * \frac{1}{\cancel{(1+z^{-1})}(1-z^{-1})^2}\Bigg|_{z=-1}$$

$$A = \frac{z^2}{(z-1)^2}\Bigg|_{z=-1} \Rightarrow A = \frac{1}{(-2)^2} = \frac{1}{4}$$

$$B_1 = \left[\frac{d}{dz}(1-z^{-1})^2 X(z)\right]_{z=1} = \left[(1-z^{-1})^2 \frac{1}{(1-z^{-1})^2(1+z^{-1})}\right]_{z=1}$$

$$= \frac{d}{dz}\frac{1}{1+z^{-1}} \Rightarrow \frac{-1 \times -1\, z^{-2}}{(1+z^{-1})^2} = \frac{z^{-2}}{(1+z^{-1})^2}\Bigg|_{z=1}$$

$$B = \frac{\cancel{z^{-2}}}{\cancel{z^{-2}}(z+1)^2}\Big|_{z=1} \Rightarrow B = \frac{1}{(1+1)^2} \Rightarrow B = \frac{1}{4}$$

$$C = (1-z^{-1})^2 X(z)\Big\}_{z=1} \Rightarrow C = \cancel{(1-z^{-1})^2}\frac{1}{\cancel{(1-z^{-1})^2}(1+z^{-1})}\Big|_{z=1}$$

$$C = \frac{z}{z+1}\Big|_{z=1} = \frac{1}{1+1} = \frac{1}{2} \Rightarrow C = \frac{1}{2}$$

the total Inverse transform is

$$X(z) = \frac{\frac{1}{4}}{1+z^{-1}} + \frac{\frac{1}{4}}{1-z^{-1}} + \frac{\frac{1}{2}}{(1-z^{-1})^2}$$

$$x(n) = \frac{1}{4}(-1)^n u(n) + \frac{1}{4}(1)^n u(n) + \frac{1}{2}(n+1)u(n)$$

$$x(n) = \frac{1}{4}\left[(-1)^n + 1 + 2(n+1)\right]u(n).$$

Q20 Find the inverse Z-transform of the second-order System

$$X(z) = \frac{1+\frac{1}{4}z^{-1}}{(1-\frac{1}{2}z^{-1})^2} \qquad |z| > \frac{1}{2}$$

$$X(z) = \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{(1-\frac{1}{2}z^{-1})^2}$$

$$A = \frac{1}{2}\left[\frac{d}{dz}(1-\tfrac{1}{2}z^{-1})^2 X(z)\right]_{z=1/2}$$

$$= \frac{1}{2}\left[\frac{d}{dz}\cancel{(1-\tfrac{1}{2}z^{-1})^2} * \frac{1+\frac{1}{4}z^{-1}}{\cancel{(1-\frac{1}{2}z^{-1})^2}}\right]\Bigg|_{z=\frac{1}{2}}$$

$$= \frac{1}{2}\left[\frac{d}{dz}(1+\tfrac{1}{4}z^{-1})\right]\Big|_{z=\frac{1}{2}} \Rightarrow \frac{1}{2}\left(\frac{-1}{4}z^{-2}\right)\Big|_{z=\frac{1}{2}}$$

$$A = \frac{-1}{8}\frac{1}{(\frac{1}{2})^2} \Rightarrow A = \frac{-1}{8}\cdot\frac{1}{\frac{1}{4}} \Rightarrow A = \frac{-1}{8} * 4$$

$$A = -\frac{1}{2}$$

$$B = \left[(1-\tfrac{1}{2}z^{-1})^2 X(z)\right]_{z=1/2}$$

$$B = \left[\cancel{(1-\tfrac{1}{2}z^{-1})^2}\frac{1+\frac{1}{4}z^{-1}}{\cancel{(1-\frac{1}{2}z^{-1})^2}}\right]_{z=\frac{1}{2}}$$

$$B = \left(1+\frac{1}{4z}\right)\Big|_{z=\frac{1}{2}} \Rightarrow B = \left(1+\frac{1}{4*\frac{1}{2}}\right)$$

$$B = 1+\frac{1}{2} \Rightarrow B = \frac{3}{2}$$

$$\therefore X(z) = \frac{1/2}{1-\frac{1}{2}z^{-1}} + \frac{3/2}{(1-\frac{1}{2}z^{-1})^2}$$

$$\therefore x(n) = -(\tfrac{1}{2})^{n+1}u(n) + 3(n+1)(\tfrac{1}{2})^{n+1}u(n)$$

Q21 Find the inverse of each of the following Z-transforms:

a) $X(z) = \ln(1-\frac{1}{2}z^{-1}) \quad |z| > \frac{1}{2}$

b) $X(z) = e^{1/z}$, with $x(n)$ a right-sided sequence

a) $$X(z) = \ln(1-\frac{1}{2}z^{-1})$$

$$\frac{d}{dz}X(z) = \frac{\frac{1}{2}z^{-2}}{1-\frac{1}{2}z^{-1}}$$

multiply both sides by $(-z)$

$$Y(z) = -z\frac{d}{dz}X(z) = \frac{\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}}$$

ROC is $|z| > \frac{1}{2}$

$$\cancel{y}(n) = -(\frac{1}{2})^n u(n-1)$$

$$y(n) = n\,x(n)$$

$$x(n) = -\frac{1}{n}(\frac{1}{2})^n u(n-1)$$

b) $x(z) = e^{1/z}$

$$\frac{d}{dz} x(z) = - e$$

Q22 Find the inverse Z-transform of $X(z) = \sin z$

by Taylor Series about $z = 0$

$$X(z) = X(z)\Big|_{z=0} + z \frac{dX(z)}{dz}\Big|_{z=0} + \frac{z^2}{2!}\frac{d^2X(z)}{dz^2}\Big|_{z=0} + \cdots$$

$$+ \frac{z^n}{n!}\frac{d^nX(z)}{dz^n}\Big|_{z=0} + \cdots$$

$$= z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$x(n) = (-1)^n \frac{1}{(2|n|+1)!} \qquad n = -1, -3, -5, \cdots$$

Q23 Find the Z-transform of the following sequences:

a) $x(n) = \left(\frac{1}{3}\right)^n u(n+3)$

b) $x(n) = \delta(n-5) + \delta(n) + 2^{n-1} u(-n)$

where $x_+(n) = \begin{cases} x(n) & n \geq 0 \\ 0 & n < 0 \end{cases}$

a) $X(z) = \dfrac{1}{1 - \frac{1}{3} z^{-1}} \qquad |z| > \frac{1}{3}$

b) $x_+(n) = \delta(n-5) + \delta(n) + 2^{-1}\delta(n)$

$X_1(z) = z^{-5} + 1 + \frac{1}{2} = 1.5 + z^{-5}$

Q24 Solve the difference equation

$$y(n) - \frac{1}{4}y(n-2) = \delta(n) \qquad n \geq 0$$

of initial conditions on $y(n)$ for $n < 0$ so that

$y(n) = 0$ for $n \geq 0$.

$$Y(z) - \frac{1}{4}\left(z^{-2}Y(z) + y(-1)z^{-1} + y(-2) \right) = 1$$

$$Y(z) - \frac{1}{4}z^{-2}Y(z) - \frac{1}{4}y(-1)z^{-1} - \frac{1}{4}y(-2) = 1$$

$$Y(z)\left[1 - \frac{1}{4}z^{-2} \right] = 1 + \frac{1}{4}y(-1) + \frac{1}{4}y(-2)$$

$$Y(z) = \frac{1 + \frac{1}{4}y(-1) + \frac{1}{4}y(-2)}{1 - \frac{1}{4}z^{-2}}$$

we have $\frac{1}{4}y(-1) = 0 \Rightarrow y(-1) = 0$

$1 + \frac{1}{4}y(-1) = 0 \Rightarrow y(-1) = -4$

Q25 Consider a system described by the difference equation

$$y(n) = y(n-1) - y(n-2) + 0.5\,x(n) + 0.5\,x(n-1)$$

Find the response of this system to the input

$$x(n) = (0.5)^n u(n)$$

with initial conditions $y(-1) = 0.75$ and $y(-2) = 0.25$.

$$y(z) = z^{-1}y(z) + y(-1) - \left[z^{-2}y(z) + z^{-1}y(-1) + y(-2)\right]$$
$$+ 0.5\,x(z) + 0.5\,z^{-1}x(z)$$

$$y(z) = z^{-1}y(z) - z^{-2}y(z) + y(-1) - z^{-1}y(-1) - y(-2)$$
$$+ 0.5\,X(z) + 0.5\,z^{-1}x(z)$$

$$y(z)\left[1 - z^{-1} + z^{-2}\right] = 0.75 - 0.75z^{-1} - 0.25 + 0.5X(z)$$
$$+ 0.5\,z^{-1}\,x(z).$$

$$x(n) = (0.5)^n u(n)$$

$$x(z) = \frac{1}{1-0.5z^{-1}}$$

$$y(z)\left[1-z^{-1}+z^{-2}\right] = 0.75 - 0.75z^{-1} - 0.25 + \frac{0.5}{1-0.5z^{-1}} + \frac{0.5z^{-1}}{1-0.5z^{-1}}$$

$$y(z) = \frac{0.5 - 0.75z^{-1}}{1-z^{-1}+z^{-2}} + \frac{0.5}{(1-z^{-1}+z^{-2})(1-0.5z^{-1})} + \frac{0.5z^{-1}}{(1-0.5z^{-1})(1-z^{-1}+z^{-2})}$$

$$Y(z) = \frac{0.5 - 0.75z^{-1}}{1 - z^{-1} + z^{-2}} + \frac{0.5 + 0.5z^{-1}}{(1 - z^{-1} + z^{-2})(1 - \frac{1}{2}z^{-1})}$$

Second term by partial fraction expansion

$$\frac{A}{1 - z^{-1} + z^{-2}} + \frac{B}{1 - \frac{1}{2}z^{-1}}$$

$$B = \cancel{(1 - \tfrac{1}{2}z^{-1})} \frac{0.5 + 0.5z^{-1}}{(1 - z^{-1} + z^{-2})\cancel{(1 - \tfrac{1}{2}z^{-1})}} \Bigg|_{z=0.5}$$

$$B = \frac{0.5 + \frac{0.5}{0.5}}{1 + \frac{1}{0.5} + \frac{1}{0.25}} = \frac{1.5}{1+2+4} = \frac{1.5}{7} =$$